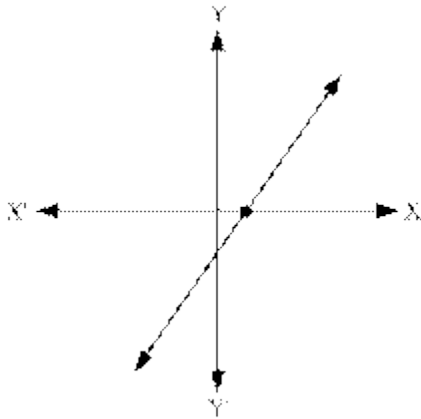


Polynomials

- **Graphic Interpretation of the number of Zeros of a Polynomial**

The zero of a polynomial, $y = p(x)$, (if it exists) is the x -coordinate of the point where the graph of $y = p(x)$ intersects the x -axis.

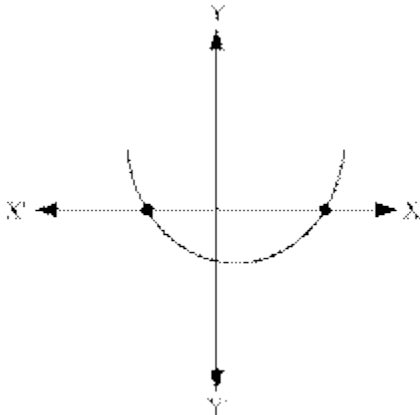
Example 1:



In the above graph, the graph intersects the x -axis at only one point.

The number of zeroes of the corresponding polynomial is 1.

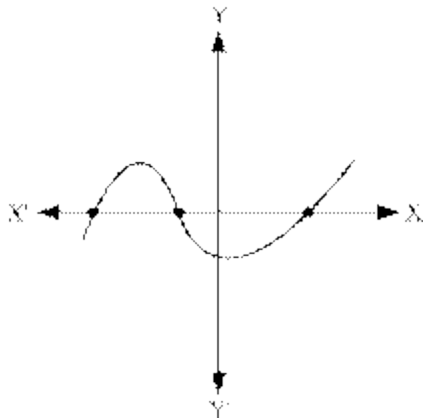
Example 2:



In the above graph, the graph intersects the x -axis at exactly two points.

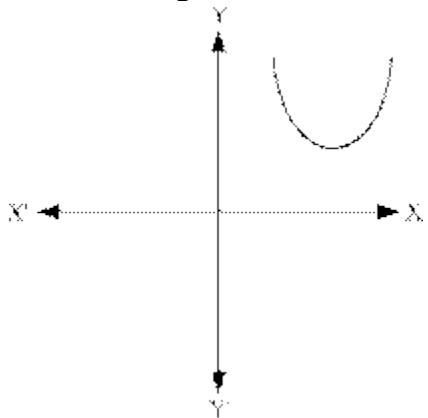
The number of zeroes of the corresponding polynomial is 2.

Example 3:



In the above graph, the graph intersects the x -axis at three points.
The number of zeroes of the corresponding polynomial is 3.

Example 4:



In the above graph, the graph does not intersect the x -axis.
The corresponding polynomial has no zeroes.

- Zeroes of a polynomial**

A real number ' k ' is a zero of a polynomial $p(x)$, if $p(k) = 0$. In this case, ' k ' is also called the root of the equation, $p(x) = 0$.

Note: A polynomial of degree n can have at most n zeroes.

Example:

- $-\frac{9}{2}$ is the zero of the linear polynomial, $2x + 9$, because $2 \times \left(-\frac{9}{2}\right) + 9 = -9 + 9 = 0$.
- 2 and -3 are the zeroes of the quadratic polynomial, $x^2 + x - 6$.

$$\left[\because 2^2 + 2 - 6 = 0, (-3)^2 + (-3) - 6 = 0 \right]$$

Example:

Find the zeroes of the polynomial, $p(x) = x^3 - 3x^2 - 6x + 8$

Solution:

By trial, we obtain

$$p(1) = 1 - 3 - 6 + 8 = 0$$

$(x - 1)$ is a factor $p(x)$ [By factor theorem]

$$p(x) = x^3 - x^2 - 2x^2 + 2x - 8x + 8$$

$$= x^2(x - 1) - 2x(x - 1) - 8(x - 1)$$

$$= (x - 1)(x^2 - 2x - 8)$$

$$= (x - 1)[x^2 - 4x + 2x - 8]$$

$$= (x - 1)[x(x - 4) + 2(x - 4)]$$

$$= (x - 1)(x - 4)(x + 2)$$

$$p(x) = 0, \text{ if } x = 1, 4, \text{ or } -2$$

Thus, the zeroes of $p(x)$ are 1, 4, and -2.

- Relationship between zeroes and Coefficients of a polynomial
- Linear Polynomial

The zero of the linear polynomial, $ax + b$, is $\frac{-b}{a} = \frac{-(\text{Constant term})}{\text{Coefficient of } x}$

Example: $3x - 5$

$$3x - 5 = 0 \Rightarrow x = \frac{5}{3}$$

$$\text{Zero of } 3x - 5 \text{ is } \frac{5}{3} = \frac{-(-5)}{3} = \frac{-(\text{Constant term})}{\text{Coefficient of } x}$$

- Quadratic Polynomial

If α and β are the zeroes of the quadratic polynomial, $p(x) = ax^2 + bx + c$, then $(x - \alpha)(x - \beta)$ are the factors of $p(x)$.

$$p(x) = ax^2 + bx + c = k[x^2 - (\alpha + \beta)x + \alpha\beta], \text{ where } k \neq 0 \text{ is constant.}$$

$$\text{Sum of zeroes} = \alpha + \beta = \frac{-b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

$$\text{Product of zeroes} = \alpha\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Example:

Find the zeroes of the quadratic polynomial, $2x^2 + 17x - 9$, and verify the relationship between the zeroes and the coefficients.

Solution:

$$\begin{aligned} p(x) &= 2x^2 + 17x - 9 \\ &= 2x^2 + 18x - x - 9 \\ &= 2x(x+9) - 1(x+9) \\ &= (x+9)(2x-1) \end{aligned}$$

The zeroes of $p(x)$ are given by,

$$\begin{aligned} p(x) &= 0 \\ \Rightarrow (x+9)(2x-1) &= 0 \\ \Rightarrow 2x-1 &= 0 \text{ or } x+9 = 0 \\ \Rightarrow x &= \frac{1}{2} \text{ or } x = -9 \end{aligned}$$

Zeroes of $p(x)$ are $\alpha = \frac{1}{2}$ and $\beta = -9$

$$\text{Sum of zeroes} = \alpha + \beta = \frac{1}{2} - 9 = \frac{-17}{2} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

$$\text{Product of zeroes} = \alpha\beta = \frac{1}{2} \times -9 = \frac{-9}{2} = -\frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

• Formation of Polynomial using the Sum and Product of Zeroes

Example:

Find a quadratic polynomial, the sum and the product of whose zeroes

$$\text{are } \frac{-14}{3} \text{ and } \frac{-5}{3}.$$

Solution:

Given that,

$$\alpha + \beta = \frac{-14}{3}, \quad \alpha\beta = \frac{-5}{3}$$

The required polynomial is given by,

$$\begin{aligned} p(x) &= k[x^2 - (\alpha + \beta)x + \alpha\beta] \\ &= k\left[x^2 - \left(\frac{-14}{3}\right)x + \left(\frac{-5}{3}\right)\right] = k\left[x^2 + \frac{14}{3}x - \frac{5}{3}\right] \end{aligned}$$

For $k = 3$,

$$p(x) = 3\left[x^2 + \frac{14}{3}x - \frac{5}{3}\right] = 3x^2 + 14x - 5$$

One of the quadratic polynomials, which fit the given condition, is $3x^2 + 14x - 5$.

• Cubic polynomial

If α, β, γ are the zeroes of the cubic

polynomial, $f(x) = ax^3 + bx^2 + cx + d$, then $(x - \alpha), (x - \beta), (x - \gamma)$ are the factors of $f(x)$.

$f(x) = ax^3 + bx^2 + cx + d = k[x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma]$, where k is a non-zero constant

$$\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$$

$$\alpha\beta\gamma = \frac{-d}{a} = -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

- Division of polynomial by polynomial of degree more than 1 can be done as follows:

Example:

Divide $x^4 - x^3 + 3x^2 - x + 3$ by $x^2 - x + 1$.

Solution:

It is given that,

Dividend = $x^4 - x^3 + 3x^2 - x + 3$, Divisor = $x^2 - x + 1$

$$\begin{array}{r}
 x^2 + 2 \\
 x^2 - x + 1 \overline{) x^4 - x^3 + 3x^2 - x + 3} \\
 \underline{x^4 - x^3 + x^2} \\
 2x^2 - x + 3 \\
 \underline{2x^2 - 2x + 2} \\
 x + 1
 \end{array}$$

- **Division Algorithm of Polynomials states that:**

Dividend = Divisor \times Quotient + Remainder

i.e., $p(x) = g(x) \times q(x) + r(x)$

Here, degree of $r(x) <$ degree of $g(x)$ and degree of $q(x) =$ degree of $p(x) -$ degree of $g(x)$

Example:

By applying division algorithm, find the quotient and remainder when $p(x) = x^4 + 3x^3 + 2x^2 + 5x - \frac{5}{2}$ is divided by $g(x) = x^3 + 2x - 1$.

Solution:

$$\begin{aligned}
 p(x) &= x^4 + 3x^3 + 2x^2 + 5x - \frac{5}{2}, g(x) = x^3 + 2x - 1 \\
 \deg p(x) &= 4, \deg g(x) = 3
 \end{aligned}$$

Degree of quotient $q(x) = 4 - 3 = 1$, and deg of remainder $r(x) < \deg g(x) = 3$

Let $q(x) = ax + b$, $r(x) = cx^2 + dx + e$

By division algorithm,

$$\begin{aligned}
 p(x) &= g(x) \times q(x) + r(x) \\
 \Rightarrow x^4 + 3x^3 + 2x^2 + 5x - \frac{5}{2} &= (ax + b)(x^3 + 2x - 1) + (cx^2 + dx + e) \\
 &= ax^4 + 2ax^2 - ax + bx^3 + 2bx - b + cx^2 + dx + e \\
 &= ax^4 + bx^3 + (2a + c)x^2 + (-a + 2b + d)x + (-b + e)
 \end{aligned}$$

Equating the coefficients of respective powers, we obtain

$$a = 1, b = 3$$

$$2a + c = 2$$

$$\Rightarrow 2 + c = 2$$

$$\Rightarrow c = 0$$

$$\begin{aligned}-a + 2b + d &= 5 \\ \Rightarrow -1 + 6 + d &= 5 \\ \Rightarrow d &= 0 \\ -b + e &= -\frac{5}{2} \\ \Rightarrow e &= 3 = \frac{1}{2}\end{aligned}$$

Quotient, $q(x) = x + 3$

Remainder, $r(x) = \frac{1}{2}$